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Entanglement in Coupled Mesoscopic Circuits

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Abstract A common two-mode squeezed state is one of the most important resources in quantum information processing of continuous variables. In this paper, a two-mode squeezed rotation-entangled state is obtained in coupled mesoscopic circuits. It is shown that instantaneously switching on the external sources may result in a two-mode squeezed state of the system, which actually arises from the coupling effect. The entanglement of the squeezed rotation-entangled state is studied. The degree of entanglement for coupled mesoscoipc circuit was calculated by using the technique of integration within an ordered product of operators. It is shown that the entanglement of the two-mode squeezed rotation-entangled state is different from that of the common squeezed state, for example, the degree of entanglement of the system, which is in the squeezed state, exist maximum value at a very low temperature.

Keywords Coupled mesoscopic circuits · Density matrix · Degree of entanglement

1 Introduction

Quantum mechanics as the most successful theory in the 20th century in explaining the foundations of our world still contains great problem of measuring of states which becomes a very paradoxical one in connection with the correlations as entangled states, in particular, when their parts are spatially separated by great distances. In recent years quantum entanglement and entangled states have been paid much attention by physicists, which led to the new branch of quantum information and quantum computation [1–3]. Entanglement of quantum states is of great importance in quantum information processing, which makes it possible to

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realize quantum cryptography, quantum communication and quantum computation. Measuring the degree of entanglement is a basic task for using it. Many quantum information protocols have extended from qubit system to the multilevel systems and the quantum systems of continuous variables [4–7]. In particular, quantum information processing of continuous variables uses a two-mode squeezed-vacuum state as a resource of entanglement. Quantum teleportation [8, 9], superdense coding [10–12] and entanglement swapping [13–15] of continuous variables can be performed by means of two-mode squeezed-vacuum states. In a two-mode squeezed vacuum state, there is close entanglement between the modes, which is highly related to the squeezing parameters. So it is useful for people to control the quantum information. However, the entanglement of a two-mode squeezed-vacuum state, which can be experimentally generated, is not so large. This is an obstacle to carry out quantum communication with high performance. For instance, a high fidelity ($F \approx 1$) of quantum states cannot be achieved in continuous variable quantum teleportation. To overcome this difficulty, several methods have been proposed by which the entanglement of a two-mode squeezed vacuum state can be enhanced with finite probability [16, 17]. These methods use the state-reduction induced by the partially destructive measurement of the photon number of a two-mode squeezed-vacuum state. The methods are sometimes called de-Gaussification.

The purpose of this letter is to propose another method by which the entanglement of a two-mode squeezed rotation-entangled state can be obtained using a coupled mesoscopic circuits with external source. And we investigate the degree of entanglement.

2 Degree of Entanglement

We introduce the definition of degree of entanglement. As is known, the degree of entanglement has many definitions for describing different entangled states, but they have the same values for the bipartite entangled states. The measure of entanglement for a pure bipartite system is now easily generalized to continuous variable and is just the Von Neumann entropy of either partial density operator of the system [18]

$$S(\rho) = -\mathrm{tr}(\rho_1 \log_2 \rho_1) = -\mathrm{tr}(\rho_2 \log_2 \rho_2), \tag{1}$$

where ρ_1 (ρ_2) is the density operator of subsystem 1 (2). S denotes the Von Neumann entropy of operator. In the following, we shall investigate the degree of entanglement for entangled state of continuous variables at a usual two-mode squeezed vacuum state. It is well known that a usual two-mode squeezed vacuum state is

$$|\Psi\rangle = \exp\{r(a_1^+ a_2^+ - a_1 a_2)\}|00\rangle. \tag{2}$$

The value of r determines the degree of squeezing. The larger r, the more the state is squeezed. a_1^+ (a_1) and a_2^+ (a_2) are the creation (annihilation) operators of the system 1 and 2. The degree of entanglement of state $|\Psi\rangle$ has been given by many authors. In this subsection, we review the degree of entanglement of a two-mode squeezed vacuum state by use of a familiar example we introduce how to calculate the degree of entanglement with the IWOP of operators [19]. The normally ordered form of two-mode squeezed operator is [20]

$$|\Psi\rangle = \operatorname{sech} r{:}\{\exp(a_1^+ a_2^+ - a_1 a_2)\tanh r\} \times \exp\{(a_1^+ a_1 + a_2^+ a_2)(\operatorname{sech} r - 1)\}{:}|00\rangle. \quad (3)$$

So we have density matrix

$$\rho_{12} = \operatorname{sech}^2 r \exp(a_1^+ a_2^+ \tanh r) \times |00\rangle\langle 00| \exp(a_1 a_2 \tanh r). \quad (4)$$

The reduced density matrix of one partite is

$$\rho_1 = \operatorname{tr}_2\{\operatorname{sech}^2 r \exp(a_1^+ a_2^+ \tanh r) \times |00\rangle\langle 00| \exp(a_1 a_2 \tanh r)\}. \quad (5)$$

By using the completeness of coherent state $|z_2\rangle$,

$$\int \frac{d^2 z_2}{\pi} |z_2\rangle\langle z_2| = 1,$$

and

$$\int \frac{d^2 z}{\pi} \exp(-|z|^2 + \sigma z) f(z^*) = f(\sigma),$$

we can obtain

$$\rho_1 = \operatorname{sech}^2 r{:} \exp(a_1^+ a_1 \tanh^2 r) \exp(-a_1^+ a_1){:}. \quad (6)$$

Because of

$$\exp(a_1^+ a_1 \tanh^2 r) = \sum_{n=0}^{\infty} (a_1^+ a_1 \tanh^2 r)^n \frac{1}{n!}, \quad (7)$$

substituting (7) into (6) we have

$$\rho_1 = \operatorname{sech}^2 r\{|0\rangle\langle 0| + \tanh^2 r|1\rangle\langle 1| + \cdots + \tanh^{2n} r|n\rangle\langle n| + \cdots\}. \quad (8)$$

Notice

$$1 + 2\tanh^2 r + \cdots + (n+1)\tanh^{2n} r = \cosh^4 r,$$

we obtain the degree of entanglement for the entangled states of continuous variables in two-mode squeezed vacuum

$$S(\rho) = \cosh^2 r \log_2(\cosh^2 r) - \sinh^2 r \log_2(\sinh^2 r). \quad (9)$$

The curve of the entanglement S of common squeezed vacuum state and the squeezing amplitude parameter r is drawn in Fig. 1. We can see that S increases with the increasing of r. But S will be zero when r reaches a certain value.

3 Preparing a Two-Mode Squeezed Rotation-Entangled State

Quantum information processing of continuous variables uses a two-mode squeezed-vacuum state as a resource of entanglement. Quantum teleportation, superdense coding and entanglement swapping of continuous variables can be performed by means of two-mode squeezed-vacuum states. It has become a research hotspot to prepare two-mode squeezed vacuum state by optics or electronics [21, 22].

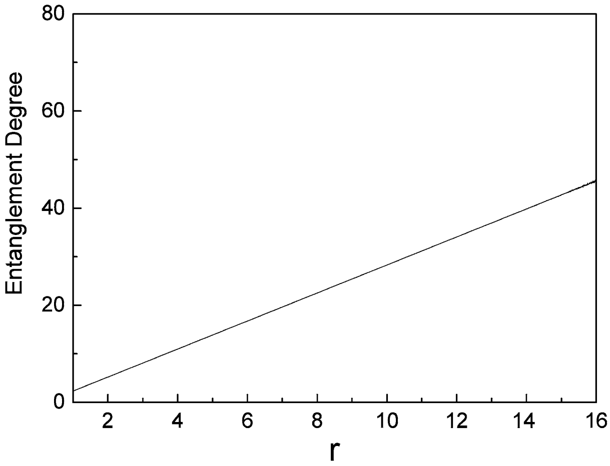


Fig. 1 The evolution of the degree of entanglement with the squeezing amplitude parameter r in a common two-mode squeezed vacuum state

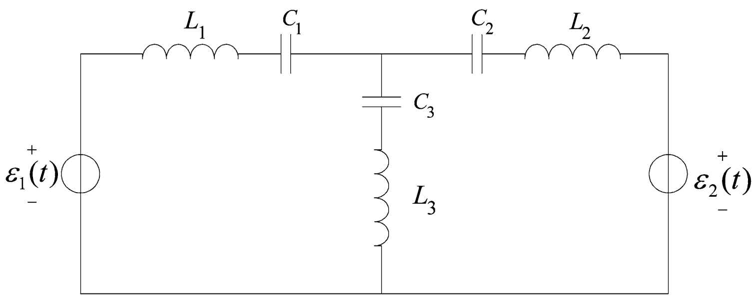


Fig. 2 Coupled circuit of inductors and capacitors

In the following, the scheme of preparing a two-mode squeezed rotation-entangled state with coupled mesoscopic circuits is put forward.

The coupled inductor and capacitor circuit is drawn in Fig. 2. When $\varepsilon_1(t) = \varepsilon_2(t) = 0$, the classical Lagrangian of this system is

$$\Xi = \frac{1}{2}[(L_1 + L_3)I_1^2 + (L_2 + L_3)I_2^2] - L_3 I_1 I_2 - \frac{1}{2}\left(\frac{C_1 + C_3}{C_1 C_3}q_1^2 + \frac{C_2 + C_3}{C_2 C_3}q_2^2\right) + \frac{1}{C_3}q_1 q_2,$$

where L_3 and C_3 stand for inductance and capacity of the coupled branch, respectively; and q_j is the electric charge. From the Lagrangian, the conjugate momenta p_j is given by

$$p_1 = \frac{\partial \Xi}{\partial \dot{q}_1} = (L_1 + L_3)I_1 - L_3 I_2,$$

$$p_2 = \frac{\partial \Xi}{\partial \dot{q}_2} = (L_2 + L_3)I_2 - L_3 I_1.$$

The variables q_j and p_j play the part of the generalized coordinate and momentum, respectively. Hence the classical Hamiltonian is

$$\begin{aligned}
&H=\frac{1}{2m_1}p_1^2+\frac{1}{2m_2}p_2^2+\frac{1}{2}m_1\omega_1^2q_1^2+\frac{1}{2}m_2\omega_2^2q_2^2+\lambda_1q_1q_2+\lambda_2p_1p_2,\\
&m_1=\frac{L_1L_2+L_2L_3+L_3L_1}{L_2+L_3},\qquad m_2=\frac{L_1L_2+L_2L_3+L_3L_1}{L_1+L_3},\\
&\omega_1^2=\frac{C_1+C_3}{m_1C_1C_3},\qquad \omega_2^2=\frac{C_2+C_3}{m_2C_2C_3},\\
&\lambda_1=-\frac{1}{C_3},\qquad \lambda_2=\frac{L_3}{L_1L_2+L_2L_3+L_3L_1}.
\end{aligned}\tag{10}$$

According to the standard quantization principle, assuming $[q_j,p_k]=i\hbar\delta_{jk}$, we can quantize the system by q_j and p_k $(j,k=1,2)$. Equation (10) represents a pair of quantized harmonic oscillators which are coupled to each other. In order to diagonalize (10), we can introduce the following unitary operator U in the coordinate representation

$$U=\int\int_{-\infty}^{+\infty}dq_1dq_2\left|\begin{pmatrix}e^r\cos\theta & e^r\sin\theta\\ -e^{-r}\sin\theta & e^r\cos\theta\end{pmatrix}\begin{pmatrix}q_1\\ q_2\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}q_1\\ q_2\end{pmatrix}\right|,\tag{11}$$

where

$$\left|\begin{pmatrix}q_1\\ q_2\end{pmatrix}\right\rangle=|q_1,q_2\rangle$$

is the two-mode coordinate eigenstate and

$$r=\ln\left[\sqrt[4]{\frac{C_1(L_2C_2-L_3C_3)}{C_2(L_1C_1-L_3C_3)}}\right],\tag{12}$$

$$\tan(2\theta)=\frac{2\sqrt{C_1C_2(L_1C_1-L_3C_3)(L_2C_2-L_3C_3)}}{L_1C_1(C_2+C_3)-L_2C_2(C_1+C_3)+L_3C_3(C_1-C_2)}.\tag{13}$$

Using the IWOP technique, we can perform the integration in (11) to get the normal produced form of U

$$\begin{aligned}
U=&\frac{1}{\pi}\int\int_{-\infty}^{+\infty}dq_1dq_2{:}\exp\left\{-\frac{1}{2}[e^{2r}(q_1\cos\theta+q_2\sin\theta)^2+e^{-2r}(q_1\sin\theta-q_2\cos\theta)^2]\right.\\
&+\sqrt{2}e^r(q_1\cos\theta+q_2\sin\theta)a_1^+ +\sqrt{2}e^{-r}(q_1\sin\theta-q_2\cos\theta)a_2^+\\
&\left.-\frac{1}{2}(q_1^2+q_2^2)+\sqrt{2}(q_1a_1+q_2a_2)-\frac{1}{2}(a_1+a_1^+)^2-\frac{1}{2}(a_2+a_2^+)^2\right\}{:}\\
=&\operatorname{sech}r\exp\left\{\frac{\tanh r}{2}(a_1^{+2}-a_2^{+2})\right\}{:}\\
&\times\exp\left\{\frac{1}{2}(a_1^+a_2^+)\begin{pmatrix}\cosh r\cos\theta-1 & \sinh r\sin\theta\\ -\sinh r\sin\theta & \cosh r\cos\theta-1\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix}\right\}{:}\\
&\times\exp\left\{\frac{\tanh r\cos 2\theta}{2}(a_2^{+2}-a_1^{+2})\right\},
\end{aligned}\tag{14}$$

where :: stands for normal product. The a_1^+ (a_1) and a_2^+ (a_2) are creation (annihilation) operators of two-mode harmonic oscillators.

$$q_j = \sqrt{\frac{\hbar}{2m_j\omega_j}}(a_j + a_j^+),$$

$$p_j = i\sqrt{\frac{\hbar m_j\omega_j}{2}}(a_j - a_j^+),$$

$j = 1, 2$. Therefore, under the U transformation a_j, becomes

$$Ua_1U^1 = (a_1\cos\theta - a_2\sin\theta)\cosh r - (a_1^+\cos\theta + a_2^+\sin\theta)\sinh r,$$

$$Ua_2U^1 = (a_1\sin\theta + a_2\cos\theta)\cosh r - (a_1^+\sin\theta - a_2^+\cos\theta)\sinh r.$$

We can see that the U transformation is a complicated one including both rotation and squeezing transformation. Although it contains squeezing parameter r and rotation parameter θ in above transformations, it can't be decomposed to two continuous and independent transformations. One is $a_1 \to (a_1\cosh r - a_1^+\sinh r)$, and the other is $a_1 \to (a_1\cos\theta + a_2\sin\theta)$. Then we call above formulas squeezed and rotated entangled transformations. Obviously when $\theta = 0$, above formulas are typical two-mode squeezed transformations.

Let the operator U with the form of a normally ordered product act on the vacuum state $|00\rangle$. Then the following expression can be obtained

$$U|00\rangle = \operatorname{sech} r \exp\left\{\frac{\tanh r}{2}(a_1^{+2} - a_2^{+2})\right\}|00\rangle. \tag{15}$$

Obviously, above formula represents a squeezed vacuum state, which is the two-mode squeezed rotation-entangled state. Because the two-mode squeezed rotation-entangled state is of different quantum characteristics with generalized squeezed vacuum state, formula (9) cannot be used to the research of entanglement for the two-mode squeezed rotation-entangled state.

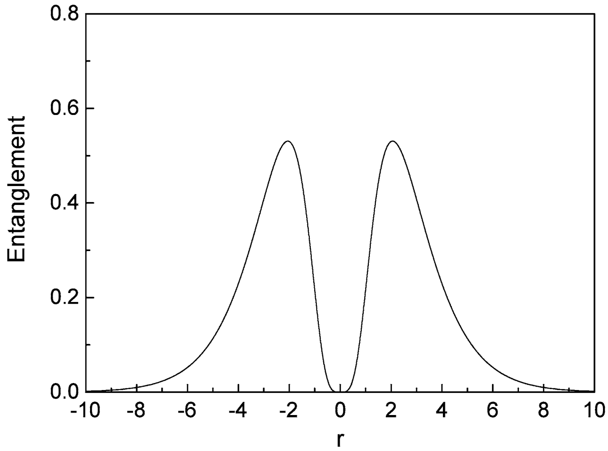
For an independent LC circuit without coupling, if it is in the vacuum state at the initial time, it will evolve to a coherent state under the effect of external sources. For the coupled mesoscopic circuit, if the system is in the vacuum state at the initial time, it will evolve to a squeezed coherent state under the effect of external sources because of the presence of the coupling. When the external electric sources are only instantaneously switched on, say, for an infinitesimal $t = \tau \to 0$ and then switched off, in this case, because of the role of coupling, the system is in squeezed state.

4 Degree of Entanglement for Coupled Mesoscopic Circuits

From (15), the density operator of the squeezed entangled state is

$$\rho_{12} = \operatorname{sech}^2 r \exp\left\{\frac{1}{2}(a_1^{+2} - a_2^{+2})\tanh r\right\}|00\rangle\langle 00|\exp\left\{\frac{1}{2}(a_1^2 - a_2^2)\tanh r\right\}. \tag{16}$$

Fig. 3 The evolution of the degree of entanglement with the squeezing amplitude parameter r in the two-mode squeezed rotation-entangled state



So, we have

$$\rho_1 = \operatorname{sech} r \colon \exp\left\{\frac{1}{2}(a_1^{+2} + a_1^2)\tanh r\right\}\exp\{-a_1^+ a_1\}\colon$$
$$= \operatorname{sech} r \sum_{n,m=0}^{\infty} \frac{\tanh^{m+n} r}{2^{m+n} m! n!} a_1^{+2n}|0\rangle\langle 0|a_1^{2m}. \tag{17}$$

Because there exists too many terms' addition in the upper equation and it is very difficult to solve it, we suppose the temperature of the system is very low and there are only two levels for each circuit, then the degree of entanglement of this system is

$$S(\rho) = -\left(\operatorname{sech} r + \frac{1}{2}\operatorname{sech} r \tanh^2 r\right)\log_2\left(\operatorname{sech} r + \frac{1}{2}\operatorname{sech} r \tanh^2 r\right). \tag{18}$$

To study the variation of degree of entanglement versus the squeezing amplitude, we have performed numerical calculations using the simulation package Mathematics. The evolution of the degree of entanglement with the squeezing amplitude r is plotted in Fig. 3.

It indicates, from (15), that the squeezing amplitude parameter is closely relative with the circuit parameters.

$$4r = \ln \frac{C_1(L_2C_2 - L_3C_3)}{C_2(L_1C_1 - L_3C_3)}. \tag{19}$$

So the degree of entanglement is relative not only with the parameters of both meshes L_j, C_j $(j = 1, 2)$ but also with the coupling parameters L_3 and C_3. When coupling of inductance or capacitance coexist and the mesh parameters are determined, r is only depend on the coupling parameter. While there is only coupling of inductance or capacitance, from (15) and (18), we can know that the degree of entanglement only relates with parameters of meshes but not the coupling parameter.

From Fig. 3, we can see that the curve is like a parabola flares downward and the degree of entanglement takes its maximum at $|r| \approx 2$, which is different form that of the common squeezed state. However, when r increases to a certain value, the degree of entanglement of both states become zero.

5 Conclusion

The two-mode squeezed rotation-entangled state can be generated by the action of an external source on initial vacuum state of a coupled mesoscopic circuits. Although the degree of entanglement of both states is relative to r, the detail relationship is different. The research in the paper will be helpful to miniaturize integrate circuits and electric components. It will be also significant for the utilization of mesoscopic circuits to evolve the quantum states, which work as quantum information carriers.

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